Pergamon

J. Appl. Maths Mechs, Vol. 57, No. 6, pp. 995–1004, 1993

0021–8928/93 $24.00+0.00

0021-8928(94)E0005-U

THE CONTROLLABILITY OF DYNAMICAL SYSTEMS WITH RESPECT TO PART OF THE VARIABLES†

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(*Received 22 April 1992*)

A particular formulation of the problem of control with respect to part of the variables, in which the initial and terminal states of the system belong to the same subspace, is considered. Necessary and sufficient conditions are established for linear autonomous systems of this type to be partially controllable, i.e. controllable with respect to part of the variables. Using the method of oriented manifolds [1], several theorems concerning the partial controllability of non-linear autonomous systems are proved. The control of the rotational motion of a rigid body by a single rotor is investigated.

THE PROPERTY of partial controllability has been used in stabilization [2, 3] and optimal-control [4] problems. In these applications the initial state of the system was assumed to be arbitrary. This formulation has been most thoroughly investigated; in particular, necessary and sufficient conditions have been obtained for linear systems [5, 6]. This paper is a sequel to [7]; the method used enables one to obtain fairly broad conditions for the controllability of non-linear systems and to reduce their verification to the analysis of a system of partial differential equations, similar to the equations for Lyapunov functions in stability theory; the results are thus readily applicable to stabilization problems.

1. STATEMENT OF THE PROBLEM

We will consider dynamical systems described by ordinary differential equations

$$\dot{x} = f(x,u), \quad x \in D \subseteq R^n, \quad u \in U \subseteq R^m, \quad t \in T = [0,\infty) \tag{1.1}$$

where x is the phase vector and u the control vector, which is a bounded measurable function of the time t. The domains D and U are assumed to be convex and to contain the origin as an interior point. The function f is assumed to be continuously differentiable a sufficient number of times.

When investigating the motion (1.1) one often encounters situations in which the system has integrals, so that it is uncontrollable (with respect to all the variables), and one must consider partial controllability. An example is the problem of controlling the rotational motion of a rigid body with a rotor, in which the absolute value of the total angular momentum of the carrier-body and the rotor is an integral, and the system is uncontrollable by varying the angular velocities of the body and the rotor. In reality, however, it is more important that the system, be controllable by varying the angular velocity of the carrier-body, while the motion of the rotor is unimportant. One thus arrives at the notion of partial controllability (in this case with respect to

†*Prikl. Mat. Mekh.* Vol. 57, No. 6, pp. 41–50, 1993.

the angular velocity of the body), which can be defined in different ways. The possibilities of control are exploited most completely if the non-distinguished variables are allowed to vary subject to no constraints whatever; that is to say, the behaviour of these variables may be selected in a special way to construct the required control.

We will divide the phase vector into two subvectors, $x^T = (x_\alpha^T,\ x_\beta^T)\ (x_\alpha \in D_\alpha \subseteq R^\alpha,\ x_\beta \in D_\beta \subseteq R^\beta)$ and introduce several definitions for system (1.1).

Definition 1. System (1.1) is controllable with respect to the variable x_α in the domain D if, for any $x_{\alpha 0},\ x_{\alpha 1} \in D_\alpha$, a time $t_1 \in T$ and an admissible control $u(t)$ exist such that the corresponding solution $x(t)$ of system (1.1) satisfies the conditions

$$x_\alpha(0) = x_{\alpha 0},\quad x_\alpha(t_1) = x_{\alpha 1},\quad x(t) \in D \quad \text{for} \quad 0 \leqslant t \leqslant t_1$$

Definition 2. System (1.1) is locally controllable (in the neighbourhood of zero) with respect to the variable x_α if closed sets $G_1 \subset D_\alpha,\ G_2 \subset D$ exist, containing the origin, such that for any $x_{\alpha 0},\ x_{\alpha 1} \in G_1$ a time $t_1 \in T$ and an admissible control $u(t)$ exist such that the corresponding solution $x(t)$ of system, (1.1) satisfies the conditions

$$x_\alpha(0) = x_{\alpha 0},\quad x_\alpha(t_1) = x_{\alpha 1},\quad x(t) \in G_2 \quad \text{for} \quad 0 \leqslant t \leqslant t_1$$

2. NECESSARY AND SUFFICIENT CONDITIONS FOR THE PARTIAL CONTROLLABILITY OF LINEAR SYSTEMS

We will first consider controllability with respect to one coordinate $x_\alpha = \alpha' x$ of a linear system

$$\dot{x} = Ax + Bu \tag{2.1}$$

on the assumption that $(B, AB, \ldots, A^{n-1}B) = r < n$. Denote the subspace of controllability by R_c.

Theorem 1. System (2.1) is controllable with respect to the coordinate $x_\alpha = \alpha^T x$ if and only if α is not an eigenvector of the matrix A^T orthogonal to the vectors $b_1,\ b_2,\ \ldots,\ b_m$, where $B = (b_1, \ldots, b_m)$.

Proof. Necessity. Suppose, contrary to the assumption, that α is an eigenvector of A^T (with eigenvalue λ) orthogonal to the vectors $b_1, \ldots, b_m$. Then, using Eqs (2.1), we get $\dot{x}_\alpha = \lambda x_\alpha$. Hence it follows that for any $t_1 \in T$ values $x_{\alpha 0},\ x_{\alpha 1}$ exist not assumed by any solution $x_\alpha(t)$ of system (2.1), i.e. the system is not controllable with respect to x_α. This contradiction proves the theorem.

Sufficiency. We will consider the solution of the boundary-value problem separately in two cases. In the first case, we assume that α is not orthogonal to the vectors $b_1, \ldots, b_m$. Transforming to a new basis whose first r vectors lie in the subspace of controllability R_c, we conclude that in the new coordinates y the coordinate x_α has the representation $x_\alpha = \tilde{\alpha}^T y$, where at least one of the numbers $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_r$ does not vanish. Then for any $x_{\alpha 0}$ and $x_{\alpha 1}$ two points $y_i = (y_{1i}, \ldots, y_{ri}, 0, \ldots, 0) \in R_c\ (i = 0, 1)$ exist such that $x_{\alpha i} = \tilde{\alpha}_1 y_{1i} + \ldots + \tilde{\alpha}_n y_{ni}$. Since $y_i \in R_c$, a control $u(t)$ and a time $t_1 \in T$ exist such that the corresponding solution $y(t)$ satisfies the conditions $y(0) = y_0,\ y(t_1) = y_1$. Then the solution of system (2.1) under the control $u(t)$ satisfies the conditions $x_\alpha(0) = x_{\alpha 0},\ x_\alpha(t_1) = x_{\alpha 1}$, proving the assertion in this case.

It remains to consider the case in which α is orthogonal to the vectors $b_1, \ldots, b_m$ but is not an eigenvector of A^T. To solve the boundary-value problem, we use the explicit form of the solution $x_\alpha(t)$. All possible cases may be reduced to the following three

1. $x_\alpha(t) = e^{\gamma t}(c_1 \cos\beta t + c_2 \sin\beta t)$
2. $x_\alpha(t) = e^{\lambda t}(c_1 t^{m-1} + c_2 t^{m-2}),\ m \geqslant 2$
3. $x_\alpha(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t},\ \lambda_1 \neq \lambda_2,\ c_i = \text{const}.$

In all three cases, whatever the values of $x_{\alpha 0}$, $x_{\alpha 1}$, one can choose constants c_i and α and a time t_i so as to satisfy the boundary condition. This proves the theorem.

Example 1. Let us examine the controllability of the system

$$\dot{x}_1 = x_1 + x_2 + u, \quad \dot{x}_2 = 2x_1 + x_2 + x_3 + u, \quad \dot{x}_3 = x_2 + x_3 + u \tag{2.2}$$

This system is uncontrollable (with respect to all the variables), since $\operatorname{rank}(b, Ab, A^2b) = 2 < 3$. Using Theorem 1, let us see whether it is controllable with respect to any coordinate, defining the vectors α_i $(i = 1, \ldots, 5)$ to be the standard unit basis vectors and the vectors $\alpha_4 = (2, -1, -1)^T$, $\alpha_5 = (-1, 0, 1)^T$. The basis vectors are not orthogonal to the vector $b = (1, 1, 1)^T$, so by Theorem 1 system (2.2) is controllable with respect to x_i $(i = 1, 2, 3)$. The vectors α_4 and α_5 are orthogonal to b and, in addition, $A^T\alpha_4 = (0, 0, -2)^T \neq \lambda\alpha_4$, $A^T\alpha_5 = (-1, 0, 1)^T = \alpha_5$; hence, by Theorem 1, the system is controllable with respect to $x_4 = 2x_1 - x_2 - x_3$ but uncontrollable with respect to $x_5 = -x_1 + x_3$.

Proceeding now to investigate the controllability of system (2.1) with respect to a variable $x_\alpha = (\alpha_1, \ldots, \alpha_k)^T x$, we shall assume that $(\alpha_1, \ldots, \alpha_k) = k$. A non-singular linear substitution reduces system (2.1) to canonical form

$$\dot{y} = Py + Qu; \quad P = \left\| \begin{matrix} P_1 & P_2 \\ 0 & P_3 \end{matrix} \right\|, \quad Q = \left\| \begin{matrix} Q_1 \\ 0 \end{matrix} \right\| \tag{2.3}$$

where P_1, P_2, P_3 and Q_1 are matrices of dimensions $r \times r$, $r \times (n-r)$, $(n-r) \times (n-r)$ and $r \times m$, respectively.

Examining the projections of the vectors $\alpha_1, \ldots, \alpha_k$ on to the subspace of controllability R_c, let us suppose that s $(s \leqslant k)$ of them are linearly independent. Then, changing if necessary by a non-singular linear transformation, to a new variable $\tilde{x}_\alpha$, we may assume that the projections of the first s vectors $\alpha_1, \ldots, \alpha_s$ are independent, while the remaining vectors $\alpha_{s+1}, \ldots, \alpha_k$ project to zero, i.e.

$$\tilde{x}_{\alpha i} = \sum_{j=r+1}^{n} \tilde{\alpha}_{ij} y_j \quad (i = s+1, \ldots, k)$$

Now, repeating the reasoning in the proof of Theorem 1, we see that system (2.3) is controllable if and only if, for any matrix $K_p = (\tilde{\alpha}_{i1}^n, \ldots, \tilde{\alpha}_{ip}^n)$ built up from p different vectors $\tilde{\alpha}_j^n = (\tilde{\alpha}_{jr+1}, \ldots, \tilde{\alpha}_{jn})^T (j = s+1, \ldots, k)$, we have

$$\operatorname{rank}(K_p, P_3^T K_p, \ldots, P_3^{T(n-r-1)} K_p) \geqslant 2p \quad (p = 1, \ldots, k-s)$$

The results may be stated as the following theorem.

Theorem 2. Let the vectors $\alpha_i = (\alpha_{il}, \ldots, \alpha_{in})^T (i = 1, \ldots, k)$ be such that $\operatorname{rank} \|\alpha_{ij}\| = s$ $(i = 1, \ldots, s;\ j = 1, \ldots, r)$, $\alpha_{ej} = 0$ $(l = s+1, \ldots, k;\ j = 1, \ldots, r)$ and $\alpha_j^n = (\alpha_{s+jr+1}, \ldots, \alpha_{s+jn})^T$ $(j = 1, \ldots, k-s)$. Then system (2.3) is controllable with respect to $x_\alpha = (\alpha_1, \ldots, \alpha_k)^T y$ if and only if, for any matrix $K_p = (\alpha_{i_1}^n, \ldots, \alpha_{i_p}^n)$ built up from p different vectors α_j^n $(j = 1, \ldots, k-s)$

$$\operatorname{rank}(K_p, P_3^T K_p, \ldots, P_3^{t(n-r-1)} K_p) \geqslant 2p \quad (p = 1, \ldots, k-s) \tag{2.4}$$

Remark 1. The conditions of Theorem 2 are clearly satisfied if the vectors $\alpha_1, \ldots, \alpha_k$ belong to the subspace of controllability R_c, i.e. if

$$\operatorname{rank}(B, AB, \ldots, A^{n-1}B) = \operatorname{rank}(\alpha_1, \ldots, \alpha_k, B, AB, \ldots, A^{n-1}B) \tag{2.5}$$

which is a sufficient condition for system (2.1) to be controllable with respect to $x_\alpha = (\alpha_1, \ldots, \alpha_k)^T x$.

Remark 2. Since $\operatorname{rank}(K_p, P_3^T K_p, \ldots, P_3^{\prime(n-r-1)} K_p) \leq n-r$, it follows from inequality (2.4) for $p=k-s$ that

$$n-r \geq 2(k-s) \tag{2.6}$$

which is a necessary condition for system (2.3) to be controllable with respect to x_α.

In particular, if $k=n-1$, inequality (2.6) and the condition $r \geq s$ imply that $r=s=n-1$ or $r=s=n-2$. In the first case the vectors $\alpha_1, \ldots, \alpha_{n-1}$ belong to the subspace of controllability, i.e. the sufficient condition is also satisfied. In the second case, one can show, as in the proof of Theorem 1, that the sufficient condition for controllability is satisfied if the vector α_{n-1}^n is not an eigenvector of the matrix P_3^T. We have thus proved the following corollary.

Corollary 1. System (2.3) is controllable with respect to $x_\alpha = (\alpha_1, \ldots, \alpha_{n-1})' y$ if and only if either $r=s=n-1$, or $r=s=n-2$ and α_{n-1}^n is not an eigenvector of P_3^T.

Example 2. Let us examine the controllability of the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = x_4, \quad \dot{x}_4 = x_5, \quad \dot{x}_5 = 0 \tag{2.7}$$

The system as a whole does not contain a control, and it is obviously uncontrollable with respect to all the variables. By the necessary condition (2.6), it may be controllable with respect to two coordinates. Applying Theorem 2, we can show that the possible coordinate pairs are (x_1, x_2), (x_1, x_3), (x_1, x_4), (x_2, x_3), (x_2, x_4). System (2.7) is uncontrollable with respect to any other pair of coordinates (x_i, x_j). It is controllable with respect to any single coordinate, with the sole exception of x_5.

Analysing the structure of the trajectories in the uncontrollable cases, we conclude that the reason for uncontrollability with respect to a variable x_α is the existence of an invariant hyperplane $\alpha^T x = 0$, where $\alpha \in R^\alpha$. Thus, in Example 1, the invariant plane is the subspace of controllability $x_3 - x_1 = 0$, which explains why the system is uncontrollable with respect to the variable $x_\alpha = x_3 - x_1$. Theorem 1 and Corollary 1 can be rephrased to state that the necessary and sufficient condition is the absence of an invariant hyperplane $\alpha^T x = 0$ $(\alpha \in R^\alpha)$; this approach enables our results to be generalized quite easily to non-linear systems.

3. CONTROLLABILITY OF NON-LINEAR SYSTEMS WITH RESPECT TO PART OF THE VARIABLES

Non-linear systems can be conveniently investigated by geometrical methods, so the above-mentioned formulation of the controllability conditions in terms of invariant hyperplanes—or, more generally, invariant manifolds (IMs)—will now be useful. However, the structure of non-linear systems may be quite complicated, and they may lack IMs entirely, admitting of only oriented manifolds (OMs) [7],† which are in this case the source of the uncontrollability.

We will illustrate this with some examples.

Example 3. Examine the controllability of the system

$$\dot{x} = y, \quad \dot{y} = y(1+\cos x) - \sin x, \quad \dot{z} = u \tag{3.1}$$

This system is in triangular form [1] and is uncontrollable in any domain, because the equations for x, y form a separate subsystem which does not contain a control. The system is obviously controllable with respect to the coordinate z. To study the controllability with respect to x and y, we need only consider

†See also KOVALEV A. M., The method of invariant relations in control theory for dynamical systems, with applications to problems of mechanics. Preprint No. 01, Inst. Prikl. Mat. i Mekh. Akad. Nauk Ukrainy, Donetsk, 1992.

system (3.1) in the xy plane. System (3.1) does not admit of the IM $x=0$, $y=0$, and the conditions of Theorem 1 are satisfied. Analysing the velocity field of the system in the xy plane, we conclude that the system is indeed locally controllable with respect to x and with respect to y. For example, a direct check, using a theorem of Levi-Civita [8], will verify that system (3.1) has the IM $y=\sin x$, whose existence implies that the system is uncontrollable with respect to y in the whole space or in a domain containing the origin, e.g. in the sphere $x^2+y^2+z^2 \leqslant R^2$ $(R>1)$. At the same time, it is controllable with respect to $x_\alpha = k_1 x + k_2 y$, where $k_1 \neq 0$, in the whole space (Fig. 1).

Example 4. Consider the system

$$\dot{x}=2y, \quad \dot{y}=3x^2, \quad \dot{z}=u \tag{3.2}$$

As in the previous case, it can be shown that the conditions of Theorem 1 hold and the system is locally controllable with respect to x and with respect to y, because there are no IMs orthogonal to the x and y axes. Analysing the velocity field, we see that system (3.2) is controllable with respect to x. With respect to y, however, the system is not controllable in any domain, because any OM $y=\text{const}$, in particular, the x axis, is orthogonal to the y axis (Fig. 2).

These examples show how, as in the theory of controllability with respect to all the variables [1], the use of IMs will yield only necessary conditions for partial controllability. To state these conditions we define sets $D_{x\alpha k}=\{x \in D : x_\alpha = x_{\alpha k}\}$—these are the sections cut out of D by the planes $x_\alpha = x_{\alpha k}$.

Theorem 3. If system (1.1) is controllable with respect to the variable x_α in the domain D, then there are in D no IMs M such that $M \supseteq D_{x\alpha 1}$ and $D \setminus M \supset D_{x\alpha 0}$ for some $x_{\alpha 0}$, $x_{\alpha 1} \in D_\alpha$.

The proof is indirect. Suppose, contrary to assumption, that an IM M exists such that $M \supseteq D_{x\alpha 1}$ and $D \setminus M \supset D_{x\alpha 0}$. Choose $x_{\alpha 0}$, $x_{\alpha 1} \in D_\alpha$ as boundary values. Then for any choice of control and values $x_{\beta 0}$, $x_{\beta 1} \in D_\beta$, system (1.1) cannot be steered in a finite time from the initial point $x_0^T = (x_{\alpha 0}^T, x_{\beta 0}^T)$ to the terminal point $x_1^T = (x_{\alpha 1}^T, x_{\beta 1}^T)$, because the initial point is not in the IM while the terminal point is; hence the latter is accessible only in infinite time. This means that system (1.1) is uncontrollable with respect to x_α in D, contrary to assumption. This proves the theorem.

Using the concept of an oriented set, we can obtain the necessary and sufficient conditions for controllability with respect to part of the variables. For the proofs, it will be convenient to rephrase Definition 1 in terms of orbits [7].

Definition 3. System (1.1) is controllable with respect to x_α in a domain D if $\mathrm{pr}_{x\alpha} \mathrm{Or}^{\pm} D_{x\alpha} = D\alpha \; \forall x_\alpha \in D_\alpha$.

Theorem 4. System (1.1) is controllable with respect to x_α in a domain D if and only if there are in D no oriented sets N such that $N \supseteq D_{x\alpha 1}$ and $D \setminus N \supset D_{x\alpha 0}$ for some $x_{\alpha 0}$, $x_{\alpha 1} \in D_\alpha$.

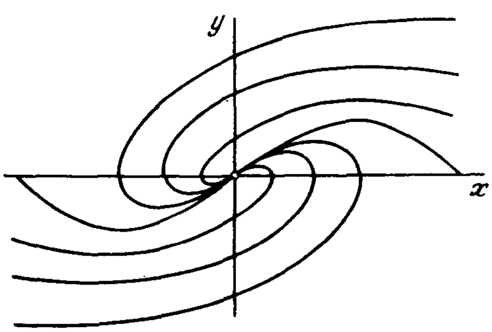

Fig. 1.

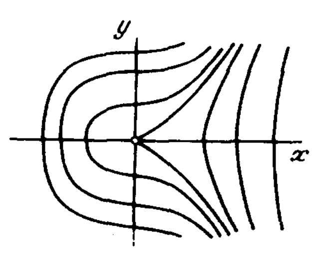

Fig. 2.

Proof. Necessity follows from Theorem 3, since an IM is a special case of an oriented set. To prove sufficiency, let us assume, on the contrary, that system (1.1) is not controllable in D, i.e. $x_\alpha \in D_\alpha$ exists such that $\mathrm{pr}_{x_\alpha}\mathrm{Or}^+D_{x\alpha} \neq D_\alpha$ or $\mathrm{pr}_{x_\alpha}\mathrm{Or}^-D_{x\alpha} \neq D_\alpha$. But $\mathrm{Or}^\pm D_{x\alpha}$ is an oriented set which clearly contains $D_{x\alpha}$, and since $\mathrm{pr}_{x_\alpha}\mathrm{Or}^\pm D_{x\alpha} \neq D_\alpha$, $x_{\alpha 0} \in D_\alpha$ exists such that $D\setminus\mathrm{Or}^-D_{x\alpha} = D_{x\alpha 0}$ or $D\setminus\mathrm{Or}^+D_{x\alpha} \supset D_{x\alpha 0}$. This contradiction completes the proof.

As to control with respect to a single coordinate $x_\alpha = \alpha^T x$, only the existence of an OM whose boundary is the hyperplane $\alpha^T x = 0$ will lead to uncontrollability. As the boundary is differentiable, the method of oriented manifolds [1, 7] yields necessary and sufficient conditions for partial controllability.

Theorem 5. System (1.1) is locally controllable with respect to a coordinate $x_\alpha = \alpha^T x$ if and only if the partial differential equation

$$(f(x,u), \nabla V(x)) = \lambda(x,u)\, V(x) + G(x,u)\ \forall u \in U \tag{3.3}$$

has no solution $V(x) = \alpha^T x$, where $\lambda(x, u)$ is a continuous function and $G(x, u)$ is a function of fixed sign in $D_0 \times U$, D_0 for some neighbourhood of the point $x = 0$.

In the consideration of control with respect to the variable $x_\alpha = (\alpha_1, \ldots, \alpha_k)^T x$, OMs that cause uncontrollability with respect to x_α have boundaries orthogonal to R^β, the subspace defined by equations $V_i(x_\alpha) = 0$, where V_i may also be non-differentiable. The method of oriented manifolds yields necessary conditions for partial controllability. As in the case of control with respect to all the variables, we distinguish between OMs of complete and incomplete dimensions [1, 7].

Theorem 6. If system (1.1) is locally controllable with respect to a variable $x_\alpha = (\alpha_1, \ldots, \alpha_k)^T x$, then the partial differential equation (3.3) has no solution $V(x) = V(x_\alpha)$.

Theorem 7. If system (1.1) is locally controllable with respect to a variable $x_\alpha = (\alpha_1, \ldots, \alpha_k)^T x$, then the system of partial differential equations

$$(f(x,u), \nabla V_i(x)) = \sum_{j=1}^{n-s} \lambda_{ij}(x,u)\, V_j(x) + G_i(x,u), \quad \forall u \in U \quad (i = 1, 2, \ldots, n-s) \tag{3.4}$$

has no solutions $V_i(x) = V_i(x_\alpha)$. Here $\lambda_{ij}(x, u)$ are continuous functions, $G_1(x, u)$ is a function of fixed sign and $G_2(x, u) = \ldots = G_{n-s}(x, u) = 0$ in $D_0 \times U$ from some neighbourhood D_0 of the point $x = 0$.

4. CONDITIONS FOR PARTIAL CONTROLLABILITY

The investigation of Eqs (3.3) and (3.4) is hindered by their inclusion of the control parameter u, which may take arbitrary values in $U \subseteq R^m$. To cope with this problem, we proceed as in the theory of controllability with respect to all the variables [1, 7], expressing the vector D at each point of $f(x, u)$ as a linear combination of vector fields $f_1(x), \ldots, f_r(x)$

$$f(x,u) = k_1(x,u)\, f_1(x) + \ldots + k_l(x,u)\, f_l(x) + k_{l+1}(x,u)\, f_{l+1}(x) + \ldots + k_r(x,u)\, f_r(x) \tag{4.1}$$

$$k_{l+1}(x,u) \geqslant 0, \ldots, k_r(x,u) \geqslant 0 \quad \forall (x,u) \in D \times U$$

where the coefficients $k_1(x, u), \ldots, k_l(x, u)$ take both positive and negative values. Then, using Theorems 5–7, one can prove the following theorems.

Theorem 8. Assume that the function $f(x, u)$ has the representation (4.1) in the domain $D \times U$. Then system (1.1) is locally controllable with respect to $x_\alpha = \alpha^T x$ if and only if the system of partial differential equations

$$(f_i(x), \nabla V(x)) = \lambda_i(x)\, V(x) + G_i(x) \quad (i = 1, \dots, r) \tag{4.2}$$

has no solution $V(x) = \alpha^T x$. Here $\lambda_i(x)$ are continuous functions, $G_j(x)$ $(j = l+1, \ . \ . \ . \ , r)$ functions of fixed sign with the same sign in D_0, and $G_1 = \ . \ . \ . \ = G_l = 0$, where D_0 is some neighbourhood of the point $x = 0$.

Theorem 9. If system (1.1) is locally controllable with respect to $x_\alpha = (\alpha_1, \ . \ . \ . \ , \alpha_k)^T x$, then the system of partial differential equations (4.2) has no solution $V(x) = V(x_\alpha)$.

Theorem 10. Assume that the function $f(x, u)$ has the representation (4.1) in the domain $D{\times}U$ and that system (1.1) is locally controllable with respect to $x_\alpha(\alpha_1, \ . \ . \ . \ , \alpha_k)^T x$. Then the system of partial differential equations

$$(f_p(x), \nabla V_i(x)) = \sum_{j=1}^{n-s} \lambda_{pij}(x)\, V_j(x) + G_{pi}(x) \tag{4.3}$$

$$(p = 1, \dots, r; \quad i = 1, \dots, n-s)$$

has no solution $V_i(x) = V_i(x_\alpha)$. Here $\lambda_{pij}(x)$ are continuous functions, $G_{p1}(x)$ $(p = l+1, \ . \ . \ . \ , r)$ are functions of fixed sign with the same sign in D_0, and $G_{pi} = 0$ for all other p, i, where D_0 is some neighbourhood of the point $x = 0$.

5. CONTROL OF THE ROTATIONAL MOTION OF A RIGID BODY

The motion of a rigid body with one rotor about its centre of mass is described by the equations [1, 9]

$$A_1\dot{\omega}_1 = (A_2 - A_3)\,\omega_2\omega_3 + (e_2\omega_3 - e_3\omega_2)\,\xi - e_1 u (123) \qquad \dot{\xi} = u \tag{5.1}$$

where ω_i and e_i are the projections on the principal axes of the angular velocity vector of the body and a unit vector in the direction of angular momentum of the rotor, ξ is the magnitude of the angular momentum of the rotor, A_i are the principal central moments of inertia of the body, and u is the moment of the forces applied to the rotor, which will be treated here as the control parameter.

Equations (5.1) may be derived from Eqs (41.9) of [9] by putting $\xi = C(\dot{\varphi} + p\alpha + qB + r\psi)$ and replacing the tensor θ_0 by a transformed tensor θ, based on representing the angular momentum vector of the body-rotor system in the form $K = \theta\omega + \xi i$, where i is the unit vector along the rotor's axis of rotation.

As Eqs (5.1) possess an integral

$$(A_1\omega_1 + \xi e_1)^2 + (A_2\omega_2 + \xi e_2)^2 + (A_3\omega_3 + \xi e_3)^2 = \text{const} \tag{5.2}$$

the system is uncontrollable (with respect to all the variables) in any domain.

In practice it is important to be able to steer the body-carrier from one rotational motion to another, so that system (5.1) should be controllable by varying the angular velocity $\omega = (\omega_1, \ \omega_2, \ \omega_3)$.

We will analyse system (5.1) for controllability with respect to the variables ω_1, ω_2 and ω_3, after settling the question of whether there exists an OS of full or partial dimensions (Theorem 9 or 10, respectively).

The right-hand sides of Eqs (5.1) obviously admit of representations (4.1), and Eqs (4.2) become

$$\frac{e_1}{A_1}\frac{\partial V}{\partial \omega_1} + \frac{e_2}{A_2}\frac{\partial V}{\partial \omega_2} + \frac{e_3}{A_3}\frac{\partial V}{\partial \omega_3} - \frac{\partial V}{\partial \xi} = \lambda_1 V$$

$$C_1 \frac{\partial V}{\partial \omega_1} + C_2 \frac{\partial V}{\partial \omega_2} + C_3 \frac{\partial V}{\partial \omega_3} = \lambda V + G(\omega_1, \omega_2, \omega_3, \xi)$$
$$\left(C_1 = \frac{1}{A_1}[(A_2 - A_3)\omega_2\omega_3 + (e_2\omega_3 - e_3\omega_2)\xi](123)\right) \tag{5.3}$$

By Theorem 9, a necessary condition for the system to be controllable with respect to ω is that system (5.3) should have no solution $V = V(\omega_1, \omega_2, \omega_3)$. Let us assume that $\omega = 0$, $\xi = 0$ is not a singular point of the surface $V = 0$, so that V can be represented in the form

$$V = (k, \omega) \quad (k = (k_1, k_2, k_3) \neq 0)$$

Substituting this function into the first equation of (5.3), we see that it satisfies the equation for $\lambda_1 \equiv 0$ and values of k_i such that

$$(\kappa, k) = 0 \quad (\kappa = (e_1 / A_1, e_2 / A_2, e_3 / A_3)) \tag{5.4}$$

To investigate the second equation of (5.3), let us choose λ_2 to have the form $\lambda_2 = q_0\xi + (q, \omega) + \ldots$ $(q = (q_1, q_2, q_3))$, where there is no free term (for a non-vanishing free term would immediately imply that G is of variable sign). Verification of the conditions of Theorem 9 then reduces to verification that the function

$$G = k_1C_1 + k_2C_2 + k_3C_3 - (q_0\xi + (q, \omega))(k, \omega) \tag{5.5}$$

has a fixed sign. The conditions for this quadratic form to be sign-definite imply that the principal minors of the coefficient matrix must vanish, $\Delta_{14} = 0$, $\Delta_{24} = 0$, $\Delta_{34} = 0$, so that we obtain a homogeneous system of linear equations in k_i

$$A_2A_3q_0k_1 - A_3e_3k_2 + A_2e_2k_3 = 0\,(1, 2, 3)$$

This system has a non-trivial solution $k_i = A_ie_i\beta$ ($\beta = \text{const}$) only if $q_0 = 0$. Substituting the values of k_i thus determined into (5.4), we obtain $e_1^2 + e_2^2 + e_3^2 = 0$, which contradicts the normalization condition $e_1^2 + e_2^2 + e_3^2 = 1$. Thus, system (5.1) does not have an OS of full dimensions (system (5.3) does not have a solution $V = V(\omega_1, \omega_2, \omega_3)$), and by Theorem 9 system (5.1) satisfies the necessary conditions for local controllability with respect to ω_1, ω_2, ω_3.

We will now examine the existence of an OS of partial dimensions, for which we use Theorem 10. It will suffice to consider the case $s = 2$ (two-dimensional OS), since if $s = 3$ it immediately follows from Eqs (5.3) that $e_1 = e_2 = e_3 = 0$, contradicting the normalization condition. Eqs (4.3) are

$$\frac{e_1}{A_1} \frac{\partial V_\alpha}{\partial \omega_1} + \frac{e_2}{A_2} \frac{\partial V_\alpha}{\partial \omega_2} + \frac{e_3}{A_3} \frac{\partial V_\alpha}{\partial \omega_3} - \frac{\partial V_\alpha}{\partial \xi} = \lambda_{\alpha 1}V_1 + \lambda_{\alpha 2}V_2$$
$$C_1 \frac{\partial V_\alpha}{\partial \omega_1} + C_2 \frac{\partial V_\alpha}{\partial \omega_2} + C_3 \frac{\partial V_\alpha}{\partial \omega_3} = \lambda_{2+\alpha 1}V_1 + \lambda_{2+\alpha 2}V_2 + \delta_{2+\alpha 3}G, \quad \alpha = 1, 2 \tag{5.6}$$

By Theorem 10, a necessary condition for system (5.1) to be controllable with respect to ω is that system (5.6) should have no solutions $V_1 = V_1(\omega_1, \omega_2, \omega_3)$, $V_2 = V_2(\omega_1, \omega_2, \omega_3)$. Let us assume that $\omega = 0$, $\xi = 0$ is a point of intersection of the surfaces $V_1 = 0$, $V_2 = 0$ but not a singular point of either surface. Then V_1 and V_2 can be represented in the form

$$V_1 = (k, \omega) + \ldots, \quad V_2 = (n, \omega) + \ldots$$

where $k=(k_1, k_2, k_3)$ and $n=(n_1, n_2, n_3)$ are non-zero and non-collinear vectors. Substituting these functions into the first two equations of (5.6), we see that these equations hold for $\lambda_{11}=\lambda_{12}=\lambda_{21}=\lambda_{22}=0$ and values of k_i, n_i that satisfy the equations

$$(\kappa, k)=0, \quad (\kappa, n)=0 \tag{5.7}$$

To investigate the other two equations of (5.6), we choose the functions λ_{ij} to have the form $\lambda_{ij}=q_{ij0}\xi+(q_{ij}, \omega)+\ldots$ $q_{ij}=(q_{ij1}, q_{ij2}, q_{ij3})$, where there are no free terms (for non-vanishing free terms would imply either that G is of variable sign or that the condition for the surfaces $V_1=0$, $V_2=0$ to intersect is violated). Then G may be written as

$$G=(k,C)-(q_{310}\xi+(q_{31},\omega))(k,\omega)-(q_{320}\xi+(q_{32},\omega))(n,\omega) \tag{5.8}$$

The condition that this function should have a fixed sign implies that the quadratic form of G is independent of ξ.

For further analysis, we use the conditions for system (5.6) to be consistent. Consider the Jacobi brackets of the first and third and the second and fourth equations

$$B_1\frac{\partial V_\alpha}{\partial\omega_1}+B_2\frac{\partial V_\alpha}{\partial\omega_2}+B_3\frac{\partial V_\alpha}{\partial\omega_3}+D_{2+\alpha1}V_1+D_{2+\alpha2}V_2+\delta_{2+\alpha3}Q=0$$

$$B_1=\frac{(A_3-A_2)}{A_1A_2A_3}(e_2A_3\omega_3+e_3A_2\omega_2+e_2e_3\xi)\,(123)$$

$$D_{2+\alpha\beta}=\frac{e_1}{A_1}\frac{\partial\lambda_{2+\alpha\beta}}{\partial\omega_1}+\frac{e_2}{A_2}\frac{\partial\lambda_{2+\alpha\beta}}{\partial\omega_2}+\frac{e_3}{A_3}\frac{\partial\lambda_{2+\alpha\beta}}{\partial\omega_3}-\frac{\partial\lambda_{2+\alpha\beta}}{\partial\xi}$$

$$Q=\frac{e_1}{A_1}\frac{\partial G}{\partial\omega_1}+\frac{e_2}{A_2}\frac{\partial G}{\partial\omega_2}+\frac{e_3}{A_3}\frac{\partial G}{\partial\omega_3}-\frac{\partial G}{\partial\xi},\quad \alpha,\beta=1,2$$

Again substituting the expressions for V_1, V_2 and taking into account that G is independent of ξ, we equate the coefficient of ξ to zero and obtain two equations

$$(\chi,k)=0,\quad (\chi,n)=0 \tag{5.9}$$
$$(\chi=((A_3-A_2)\,e_2e_3,(A_1-A_3)\,e_1e_3,(A_2-A_1)\,e_2e_1))$$

Considering Eqs (5.7) and (5.9) simultaneously, we conclude that a necessary condition for the vectors k and n to be non-collinear is that the vectors κ and χ should be collinear.

Equating the vector product of these vectors to zero, we obtain a system of equations whose solution is

$$e_2=e_3=0,\quad e_1=1\ (123) \tag{5.10}$$

A direct check will verify that, under conditions (5.10), system (5.6) has a solution

$$V_1=\omega_2,\quad V_2=\omega_3$$
$$\lambda_{11}=\lambda_{12}=\lambda_{21}=\lambda_{22}=\lambda_{31}=\lambda_{42}=0,\quad G=0$$
$$\lambda_{32}=-\frac{e_1}{A_2}\xi+\frac{A_3-A_1}{A_2}\omega_1,\quad \lambda_{41}=\frac{e_1}{A_3}\xi+\frac{A_1-A_2}{A_3}\omega_1$$

Thus, it follows from Theorem 10 that under conditions (5.10) system (5.1) is uncontrollable with respect to the variables ω_1, ω_2, ω_3; if the parameters e_i do not satisfy conditions (5.10), the system satisfies the necessary condition for local controllability with respect to ω_1, ω_2, ω_3.

In conclusion, we note that the use of a rotor to control the rotational motion of a rigid body offers more possibilities for control than the use of a jet engine, for which there are additional forbidden positions relative to the body-carrier [1, 10].

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Translated by D.L.